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Frictional coastal trapped waves in a two-layered ocean

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The effects of bottom friction on coastal trapped waves were investigated using an f-plane, two-layer model including shelf-slope topography. At a change-over latitude where the phase speeds of the internal Kelvin wave and the continental-shelf wave coincide, there are two types of behaviour of the 'frictional' eigenvalue (the phase speed and the damping rate) and the eigenfunctions, in terms of the inertial frequency f: if the frequency ω is large, wave characteristics change from one wave to another with f (Case I); while if ω is small enough, the wave characteristics do not change (Case II). In actual environments, it is predicted that the weather-band phenomena (period 2 days to 2 weeks) correspond to Case I, and very low-frequency (VLF) phenomena such as signals of El Niño along the American Continent correspond to Case II. Further, for baroclinic VLF waves, it is found that bottom friction retards the lower-layer velocity, which causes a decrease in damping. Therefore, the VLF signals caused by El Niño can travel far from their origin, overcoming the effect of bottom friction. A bottom-intensified structure in barotropic VLF waves, due to bottom friction, has also been found.

1. Introduction

Low-frequency fluctuations in the coastal ocean have frequently been investigated using the theory of coastal trapped waves. For example, models have succeeded in reproducing observations of, especially, sea-level fluctuations (e.g. Battisti & Hickey 1984; Church *et al.* 1986). In these models, which describe wave propagation in actual environments, bottom friction plays an important role because it is the principal dissipation mechanism for coastal trapped waves.

For weather-band fluctuations (period 2 days–2 weeks), the model formulation including the effects of bottom friction is now well established for very generalized models including continuous stratification and arbitrary bottom topography (Clarke & Van Gorder 1986). Mathematically, the weather band means that the typical damping timescale D/r is longer than $1/\omega$, where r is a friction coefficient, D the typical depth over the shelf and ω a wave frequency. Therefore, we use a perturbation method using the small parameter $r/D\omega$, which is required for convergence of the friction terms.

On the other hand, the effects of bottom friction on very low-frequency (VLF) waves have not been well investigated. It is likely that the VLF sea-level fluctuations, such as signals caused by El Niño, propagate far from their origin. This suggests that energy dissipation due to bottom friction might not be important for propagation of the VLF waves. For example, Huyer & Smith (1985) have detected the 1982/83

El Niño signals at sea level along the coast of Oregon. It had arrived by an oceanic path because the sea-level anomaly led the associated atmospheric anomalies (both local and basin-wide scale) by 2–3 months. Further, the estimated phase speed of 140 km/day is consistent with that of the internal Kelvin wave. As a result, the internal Kelvin wave was proposed as a possible mechanism for the propagation of the El Niño signal. If so, the wave propagated from the Equator along the west coast of the American Continent as far as 8000 km overcoming the damping due to, for example, bottom friction and/or energy leakage by Rossby waves. Regarding bottom friction, Allen (1984), using a two-layer model with a flat bottom, noted that the lower-layer velocity for the VLF waves can be retarded by bottom friction which, in turn, tends to reduce the effect of bottom friction. In his model, however, only the internal Kelvin wave can exist. Further development, including shelf-slope topography, is needed.

In the present paper, we discuss the effects of bottom friction on coastal trapped waves as a function of frequency. We use a two-layer model including shelf-slope topography and obtain analytical solutions. We examine the following two cases. First, in §4, bottom friction is as strong as the combined effect of stratification and topography; the combined effect means that interfacial motion over a slope produces torque for vorticity motion, while divergence due to vorticity motion produces interfacial motion (Allen 1975). In particular, we are interested in the behaviour of the phase speed and the damping rate near the change-over latitude where the wave properties (either the internal Kelvin waves or the continental-shelf waves) change from one wave to another if bottom friction is not considered (Allen & Romea 1980, hereinafter referred to as AR). Secondly, in §5, the effects on the VLF waves in which $r/\omega D$ is very large is examined. Results obtained in §5 correspond to the asymptotic results of §4. Further, the results are not restricted to those in the neighbourhood of the change-over latitude. Though the model in these sections is highly idealized, we expect that the results would be realized in continuous-stratification models and a real ocean (§6).

We use an f-plane model. Therefore, this study is not directly applicable to the poleward propagation of coastal trapped waves; the f-plane model does not include the leakage of energy flux due to Rossby waves and non-slowly varying effects. The latter implies that the energy flux is not conserved in a mode that changes continuously with latitude when the wavelength is longer than the typical lengthscale of the changes in the wave structures. Instead, our purpose is to undertand what kind of waves can overcome damping due to bottom friction and can travel far from their origin. The poleward propagation with bottom friction is discussed in a forthcoming paper from the present authors, and Mitsudera (1986).

The results obtained in this paper have been generalized to a dynamical system of linearly coupled damped oscillators (Grimshaw & Allen 1982). Therefore, there may be various phenomena, such as the coupling between planetary and topographic Rossby waves (Takeda 1985) and that between atmospheric edge and internal gravity waves (Garrett 1969), in which we expect qualitatively the same results as those in this paper.

2. Formulation

We consider a two-layered ocean on a rotating frame with an angular velocity $\frac{1}{2}f$, where f is constant. Without loss of generality, f is assumed to be positive. We used a Cartesian coodinate system (x, y, z), where z is positive upwards, x is the cross-shelf

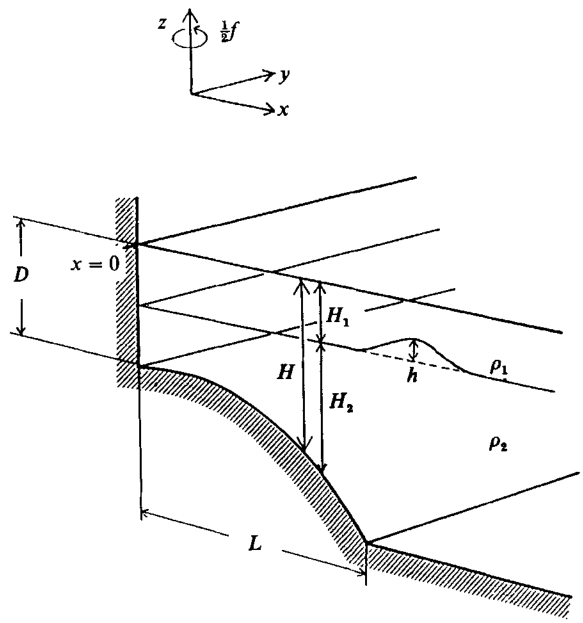

FIGURE 1. Nomenclature and geometry used in the present study.

direction and positive offshore and y is the alongshore direction. Velocity components are defined by (u_i, v_i, w_i), where subscripts $i = 1, 2$ denote variables in the upper and lower layers respectively.

Figure 1 shows the geometry considered in this study. The upper layer has density ρ_1 and a constant undisturbed thickness H_1 where the top surface is assumed to be rigid; the lower layer has density ρ_2 and a undisturbed thickness $H_2(x)$ which is independent of y. The total thickness is $H(x)\,(=H_1+H_2)$. A straight coast, at $x = 0$, has a vertical wall whose depth is D, and the interface intersects the vertical coast.

The variables are non-dimensionalized in the following manner, where the primed ones are dimensionless:

$$(x', y') = L^{-1}(x, y), \quad z' = D^{-1}z, \quad t' = ft, \quad (u_i', v_i') = U^{-1}(u_i, v_i), \quad w_i' = \frac{L}{UD} w_i,$$

$$(H_1', H_2', H') = D^{-1}(H_1, H_2, H), \quad h' = \left(\frac{\Delta\rho g}{\rho_2 f L U}\right) h,$$

$$p_1' = \frac{p_1 + \rho_1 g z}{\rho_1 f L U}, \quad p_2' = \frac{p_2 - \rho_1 g H_1 + \rho_2 g (H_1 + z)}{\rho_2 f L U}, \quad f' = \frac{f}{f_0},$$

where L is a characteristic horizontal scale, U a characteristic horizontal velocity and f_0 the Coriolis parameter at a change-over latitude. The variable p_i is the pressure, h the deviation from the undisturbed interface and t the time. The constant $\Delta\rho\,(=\rho_2 - \rho_1 \ll \rho_2)$ is the density difference between the two layers and g the acceleration due to gravity.

Henceforth, the primes are dropped from the non-dimensional variables for the sake of neatness. Here, we adopt the following long-wave approximations: (i) motions are hydrostatic; (ii) the alongshore scale of motions is much longer than the cross-shelf scale. The latter condition implies that the non-dimensional frequencies of the coastal trapped waves considered here are much smaller than unity. From these assumptions, the equations of motion and continuity become

$$(H_1 u_1)_x + (H_1 v_1)_y = S^{-1} h_t, \tag{2.1a}$$

$$-f v_1 = -p_{1x}, \quad v_{1t} + f u_1 = -p_{1y}, \tag{2.1b, c}$$

$$(H_2 u_2)_x + (H_2 v_2)_y = -S^{-1} h_t + \frac{E_{\mathrm{v}}^{\frac{1}{2}}}{2} v_{2x}, \tag{2.1d}$$

$$-f v_2 = -p_{2y}, \quad v_{2t} + f u_2 = -p_{2y}, \tag{2.1e, f}$$

where S is the stratification parameter $[= (g\Delta\rho/\rho_2)\, D/f_0^2 L^2]$ and E_{v} the vertical Ekman number $(= 4r^2/f_0^2 D^2)$. Note that the second term on the right-hand side of (2.1d) is the vertical velocity due to the Ekman pumping for low-frequency coastal trapped waves (Mitsudera & Hanawa 1987). Adding (2.1a) to (2.1d), we can define a stream function corresponding to the total transport such that

$$-\psi_y = u_1 + a^{-1} u_2 - \frac{E_{\mathrm{v}}^{\frac{1}{2}}}{2H_1} v_2, \tag{2.2a}$$

$$\psi_x = v_1 + a^{-1} v_2, \tag{2.2b}$$

where $a = H_1/H_2$. Further, continuity of pressure at the interface yields

$$h = p_2 - p_1. \tag{2.3}$$

This represents baroclinic motions. Then the equations in terms of ψ and h, which represent the vorticity equations for total transport and the velocity difference, respectively, become

$$\left(\frac{\psi_x}{H}\right)_{xt} + \frac{H_x}{H^2}(f\psi_y + h_y) = -\frac{E_{\mathrm{v}}^{\frac{1}{2}}}{2}\left[\frac{1}{H^2}(f\psi_x + h_x)\right]_x, \tag{2.4a}$$

$$f^{-1}\left[\left(\frac{h_x}{aH}\right)_x - \frac{f^2}{SH_1^2} h\right]_t + \frac{H_x}{H^2}(f\psi_y + h_y) = -\frac{E_{\mathrm{v}}^{\frac{1}{2}}}{2}\left[\frac{1}{H^2}(f\psi_x + h_x)\right]_x, \tag{2.4b}$$

with the boundary conditions

$$\psi_y = 0, \quad h_y + h_{xt} = -\frac{E_{\mathrm{v}}^{\frac{1}{2}}}{2}\frac{a}{H} f(f\psi_x + h_x) \quad \text{at} \quad x = 0, \tag{2.5a}$$

$$\psi_x, h_x \to 0 \quad \text{as} \quad x \to \infty. \tag{2.5b}$$

The boundary condition (2.5a) represents no net mass flux across the coast. The variables ψ and h can be separated into two parts such that

$$\begin{bmatrix} \psi \\ h \end{bmatrix} = \sum_{n-1}^{\infty} Q_n(y,t) \begin{bmatrix} \phi_n(x) \\ \eta_n(x) \end{bmatrix}. \tag{2.6}$$

Here, ϕ_n and η_n satisfy with the following eigenvalue problem:

$$\left(\frac{\phi_{nx}}{H}\right)_x - \frac{H_x}{\chi_n H^2}(f\phi_n + \eta_n) = 0, \tag{2.7a}$$

$$f^{-1}\left[\left(\frac{\eta_{nx}}{aH}\right)_x - \frac{f^2}{SH_1^2}\eta_n\right] - \frac{H_x}{\chi_n H^2}(f\phi_n + \eta_n) = 0, \tag{2.7b}$$

with the boundary conditions

$$\phi_n(0) = \eta_{nx}(0) - \frac{f}{\chi_n}\eta_n(0) = 0, \tag{2.8a}$$

$$\phi_{nx}(\infty) = \eta_{nx}(\infty) = 0. \tag{2.8b}$$

From (2.7) and (2.8), the orthogonality condition of the eigenfunctions becomes

$$\int_0^\infty \frac{H_x}{H^2}(f\phi_n + \eta_n)(f\phi_m + \eta_m)\,\mathrm{d}x + \frac{\eta_n(0)\,\eta_m(0)}{a(0)\,H(0)} = \delta_{nm}. \tag{2.9}$$

Equations (2.7) and (2.8) are the same as AR's (3.2)–(3.6). Further, (2.9) corresponds to AR's (3.9). Since these equations do not include the friction terms, we call χ_n a 'conservative' eigenvalue. As in AR χ_n, which represents a continental-shelf wave phase speed, increases as f increases, while that representing an internal Kelvin wave is independent of f. Further, in the neighbourhood of a change-over latitude, the conservative eigenvalues change their properties from one to another. AR's (3.28) and (3.29) represent the above characteristics as follows:

$$\tilde{\chi} = \tfrac{1}{2}\tilde{f} \pm \tfrac{1}{2}(\tilde{f}^2 + K)^{\frac{1}{2}}, \tag{2.10}$$

where $\tilde{\chi}$ and $\tilde{f}$ are the deviations from the conservative eigenvalue and the inertial frequency at the change-over latitude, respectively. These are defined by

$$\chi = -\delta_{\mathrm{R}}(1+\tilde{\chi}), \quad f = 1 + \tilde{f},$$

where δ_{R} is the non-dimensional phase speed of the internal Kelvin wave without considering the continental shelf and slope and is independent of f. The parameter K represents the coupling between topography and stratification as mentioned in the Introduction. In a weak-slope model, for example, K is represented as follows:

$$K = 8a(0)\,b_{\mathrm{s}}^2\,\delta_{\mathrm{R}}^3,$$

where topography is defined as

$$H = \begin{cases} b_{\mathrm{s}}(x-1)+1, & 0 \leqslant x < 1 \\ 1 \quad , & x \geqslant 1 \end{cases},$$

and $b_{\mathrm{s}} \ll 1$. This is the same as (3.29*b*) of AR in which $b_{\mathrm{s}} = \delta_{\mathrm{b}}^{-1}$ and $1 = f_{\mathrm{c}}$.

Substituting (2.6), (2.7) and (2.8) into (2.4) and (2.5), and considering the orthogonality condition (2.9), we may obtain equations for Q_n such that

$$\frac{1}{\chi_n}Q_{nt} + Q_{ny} = -\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2}\sum_{m=1}^{\infty} a_{mn}\,Q_m, \tag{2.11}$$

where $$a_{mn} = -\int_0^\infty \frac{1}{H^2}(f\phi_{mx} + \eta_{mx})(f\phi_{nx} + \eta_{nx})\,\mathrm{d}x.$$

If $E_{\mathrm{V}}^{\frac{1}{2}} = 0$, (2.11) represents the conservation of energy flux of each mode. The right-hand side of (2.11) are the bottom friction terms. These terms with $m \neq n$ represent the offshore and vertical phase shifts as in Brink & Allen (1978) and Brink (1982*a*). Further, these cause significant changes in the phase speed, the damping rate and the vertical profile of velocities. This will be discussed in the following sections.

3. Expansion away from the change-over latitude

In the following sections, we discuss: (i) the dependence of the 'frictional' eigenvalues (the phase speed and the damping rate) on the latitude, i.e. f; (ii) the modifications in the vertical profile of the velocities due to bottom friction, or the structure of the 'frictional' eigenfunctions. Let us consider a wavetrain solution such as

$$Q_n = \sum_{j=1}^{\infty} R_n^{(j)} \, \mathrm{e}^{\mathrm{i}\omega(c_j^{-1} y - t)}. \tag{3.1}$$

Substituting (3.1) into (2.11) yields

$$(\chi_n^{-1} - c_j^{-1}) R_n^{(j)} + \mathrm{i}\frac{E_\mathrm{V}^{\frac{1}{2}}}{2\omega} \sum_{m=1}^{\infty} a_{mn} R_m^{(j)} = 0. \tag{3.2}$$

At first, we assume that $|E_\mathrm{V}^{\frac{1}{2}}/2\omega| \ll 1$ for the sake of analytical simplicity. Although this condition imposes severe restrictions on the frictional terms, we shall obtain a solution whose structure, phase speed and damping rate are strongly affected by bottom friction as shown in §4, where the effects of bottom friction around the change-over latitude are examined.

When no two phase speeds are in the neighbourhood of the change-over latitude, the 'frictional' eigenvalue and eigenvector become

$$\frac{1}{c_j} = \frac{1}{\chi_j} + \mathrm{i}\frac{E_\mathrm{V}^{\frac{1}{2}}}{2\omega} a_{jj} + O\left(\frac{E_\mathrm{V}}{4\omega^2}\right), \tag{3.3}$$

$$R_n^{(j)} = -\mathrm{i}\frac{E_\mathrm{V}^{\frac{1}{2}}}{2\omega} \frac{R_j^{(j)} a_{nj}}{\chi_n^{-1} - \chi_j^{-1}} + O\left(\frac{E_\mathrm{V}}{4\omega^2}\right). \tag{3.4}$$

For the jth-mode wave whose amplitude is $R_j^{(j)}$, we can see from (3.4) that $R_n^{(j)}$ $(n \neq j)$ is purely imaginary, which implies that they represent the offshore and vertical phase shifts as in Brink & Allen (1978) and Brink (1982*a*). Now, we estimate c_j assuming $a(0) \ll 1$. This expansion was adopted by AR, in which we may find distinct internal Kelvin waves and continental-shelf waves away from the change-over latitude. Although the results in this section are the same as those of Mitsudera & Hanawa (1988, hereinafter referred to as MH), we shall present a brief review for completeness, as the results are needed for §4.

3.1. *The continental-shelf wave*

When variables are expanded in terms of $a(0)$ such that

$$\phi_n = \hat{\phi}_{0n} + \dots, \quad \eta_n = a(0)\,\hat{\eta}_{1n} + \dots, \quad \chi_n = \hat{\chi}_{0n} + \dots,$$

the lowest orders of (2.7) become

$$\left(\frac{\hat{\phi}_{0nx}}{H}\right)_x - \frac{fH_x}{\hat{\chi}_{0n} H^2} \hat{\phi}_{0n} = 0, \tag{3.5a}$$

$$\hat{\eta}_{1nxx} - \frac{f^2}{SH_1} \hat{\eta}_{1n} = f^2 \frac{H_x}{\chi_{0n} H} \hat{\phi}_{0n}. \tag{3.5b}$$

This represents the continental shelf wave. The real part of the frictional eigenvalue c_n becomes

$$\mathrm{Re}\,(c_n) = \chi_n = -f\frac{\int_0^\infty \frac{H_x}{H^2} f^2\phi_{0n}^2\,\mathrm{d}x}{\int_0^\infty \frac{1}{H} f^2\phi_{0nx}^2\,\mathrm{d}x} + O(a(0)). \tag{3.6}$$

Note that the phase speed of a continental-shelf wave increases as f increases because the restoring force, the topographic β-effect, becomes stronger. For the weak-slope model, for example,

$$\chi_n = -fb_{\mathrm{s}}/(n-\tfrac{1}{2})^2\pi^2 + O(a(0)) \quad (n = 1, 2, \ldots).$$

On the other hand, the imaginary part of c_n, the damping rate, becomes

$$\mathrm{Im}\left(\frac{1}{c_n}\right) = \frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega} a_{\mathrm{S}n} = -\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}\left[\int_0^\infty \frac{1}{H^2}(f\hat{\phi}_{0nx})^2\,\mathrm{d}x + O(a(0))\right], \tag{3.7}$$

which is $O(1)$ with respect to $a(0)$. For the weak-slope model, the damping rate becomes

$$\mathrm{Im}\left(\frac{1}{c_n}\right) = -\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega} b_{\mathrm{s}}^{-1}[(n-\tfrac{1}{2})^2\pi^2 + O(a(0))] \quad (n = 1, 2, \ldots).$$

3.2. *The internal Kelvin wave*

Using

$$\phi = \bar{\phi}_0 + \ldots, \quad \eta = \bar{\eta}_0 + \ldots, \quad \chi = \bar{\chi}_0 + \ldots,$$

as the expansion, the lowest-order equations with respect to $a(0)$ are

$$\left(\frac{\bar{\phi}_{0x}}{H}\right)_x - \frac{f}{\bar{\chi}_0}\frac{H_x}{H^2}\bar{\phi}_0 = \frac{1}{\bar{\chi}_0}\frac{H_x}{H^2}\bar{\eta}_0, \tag{3.8a}$$

$$\bar{\eta}_{0xx} - \frac{f^2}{SH_1}\bar{\eta}_0 = 0. \tag{3.8b}$$

Equation (3.8*b*) represents an internal Kelvin wave whose structure is

$$\bar{\eta}_0 = \bar{\eta}_0(0)\,\mathrm{e}^{(-f/\delta_{\mathrm{R}})x},$$

where $\bar{\eta}_0(0) = O(a(0)^{\frac{1}{2}})$ from (2.9), and where $\delta_{\mathrm{R}} = (SH_1)^{\frac{1}{2}}$. Substituting this into (2.8*a*), we find that

$$\bar{\chi}_0 = -\delta_{\mathrm{R}}. \tag{3.9}$$

Therefore, $\bar{\chi}_0$ is independent of f. Further, (3.8*a*) and (2.8*a*) shows that $\bar{\phi}_0$ is also $O(a(0)^{\frac{1}{2}})$, which implies that

$$\mathrm{Im}\left(\frac{1}{c}\right) = \frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega} a_{\mathrm{K}} = -\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega} O(a(0)). \tag{3.10}$$

For example, the weak-slope model shows

$$\mathrm{Im}\left(\frac{1}{c}\right) = -\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}\frac{f}{2\delta_{\mathrm{R}}} a(0).$$

3.3. *Summary*

So far, we have derived the typical scales for the phase speed (3.6) and (3.9), and those for the damping rate (3.7) and (3.10). Since the conservative eigenvalues and eigenfunctions change their characteristics from one wave to another in the neighbourhood of the change-over latitude, the asymptotes of a_{11} and a_{22} around there, where subscripts 1 and 2 denote the two adjacent modes, become as follows:

$$a_{11,22} \approx \begin{cases} a_{\mathrm{K,S}} & \text{as} \quad \tilde{f} \to -\infty \\ a_{\mathrm{S,K}} & \text{as} \quad \tilde{f} \to \infty. \end{cases} \tag{3.11}$$

A detailed discussion is given in MH.

Comparing (3.10) with (3.7), we find that the damping of the internal Kelvin wave is weaker than that of the continental-shelf wave, though the differential dissipation obtained here is dependent on the assumption that $a(0)$ is very small. However, the qualitative result is valid even when $a(0) \geqslant 1$ (as shown by numerical calculations in MH); the bottom slope causes a surface intensified structure in the internal Kelvin wave, which decreases the effects of bottom friction, while stratification causes a bottom intensified structure in the continental-shelf wave. Further, in the continuous-stratification models, both surface and bottom-intensified waves probably exist at low latitudes (e.g. Brink 1982*b*). Therefore, such differential dissipation is likely to be realized in a real ocean.

4. Expansion in the neighbourhood of the change-over latitude

4.1. *Frictional eigenvalues: the phase speed and the damping rate*

The frictional coupling can occur at the lowest order in the neighbourhood of the change-over latitude even if $|E_{\mathrm{V}}^{\frac{1}{2}}/2\omega| \ll 1$. That is, the structure, phase speed and damping are significantly influenced when strong frictional coupling is considered.

If we assume that $K^{\frac{1}{2}}$ in (2.10) is $O(|E_{\mathrm{V}}^{\frac{1}{2}}/2\omega|)$, the lowest-order terms in (3.2) become

$$(\chi_1^{-1}-c^{-1})\,R_1+\mathrm{i}\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}a_{11}\,R_1+\mathrm{i}\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}a_{21}\,R_2 = 0,$$

$$(\chi_2^{-1}-c^{-1})\,R_2+\mathrm{i}\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}a_{22}\,R_2+\mathrm{i}\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}a_{12}\,R_1 = 0$$

for the two adjacent frictional modes at the change-over latitude. The 'mode' used here is a sequence of the frictional eigenvalues depending continuously on the latitude. Solutions for the other frictional modes are the same as (3.3) and (3.4). Considering (2.10), the frictional eigenvalues of the adjacent modes become

$$\frac{1}{c_{1,2}} = -\frac{1}{\delta_{\mathrm{R}}}+\frac{\tilde{f}}{2\delta_{\mathrm{R}}}+\mathrm{i}\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{4\omega}(a_{11}+a_{22})\pm\tfrac{1}{2}(\lambda^2-\lambda_{\mathrm{c}}^2)^{\frac{1}{2}}, \tag{4.1}$$

where $$\lambda = \frac{1}{\delta_{\mathrm{R}}}(\tilde{f}^2+K)^{\frac{1}{2}}+\mathrm{i}\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}(a_{11}-a_{22}), \quad \lambda_{\mathrm{c}}^2 = 4\left(\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}\right)^2 a_{12}\,a_{21}.$$

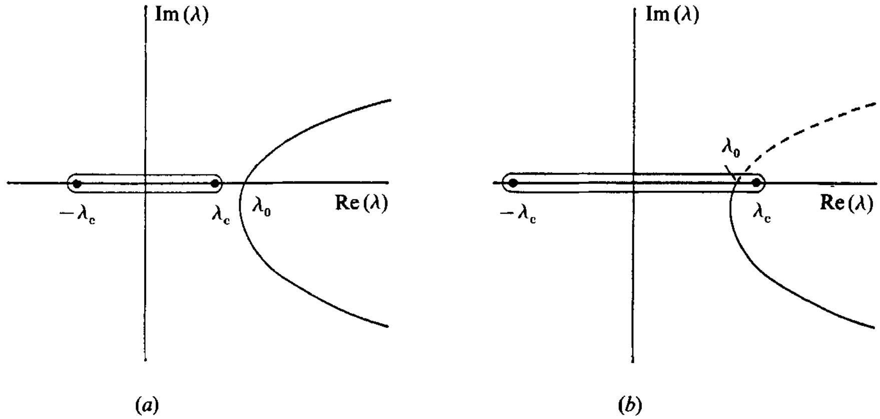

FIGURE 2. Path of λ with varying $\tilde{f}$ on the upper Riemann surface of the complex λ-plane. Branch cuts are chosen so that $\mathrm{Re}\,((\lambda^2-\lambda_c^2)^{\frac{1}{2}}) = 0$, where $\lambda_c^2 = (E_v/\omega^2)\,a_{12}a_{21}$. (*a*) Case I, and (*b*) Case II. The portion of a broken line is on the lower Riemann surface (Case II).

We can see from (4.1) that the complex λ-plane has two branch points at $\lambda = \pm\lambda_c$. We choose a branch cut so that $|\lambda| < \lambda_c$ on the real axis and define the upper (lower) Riemann surface if $\mathrm{Re}\,[(\lambda^2-\lambda_c^2)^{\frac{1}{2}}]$ is positive (negative). Since

$$\begin{cases} a_{11} < a_{22} & \text{as} \quad \tilde{f} \to -\infty \\ a_{11} > a_{22} & \text{as} \quad \tilde{f} \to \infty, \end{cases}$$

there are two possibilities as $\tilde{f}$ increases, as shown in figure 2: the path of λ remains on the upper Riemann surface if $\lambda_0 > \lambda_c$ (Case I), where λ_0 is λ at $a_{11} = a_{22}$, because $(\lambda^2-\lambda_c^2) \to \lambda$ as $\tilde{f} \to \infty$; the path resides on the lower Riemann surface if $\lambda_0 < \lambda_c$ (Case II) because $(\lambda^2-\lambda_c^2)^{\frac{1}{2}} \to -\lambda$ as $\tilde{f} \to \infty$. Therefore, the asymptotes of the 'frictional' eigenvalues $c_{1,2}$ are

$$\left.\begin{aligned} \frac{1}{c_{1,2}} &\approx \frac{1}{\chi_{1,2}} + \mathrm{i}\frac{E_v^{\frac{1}{2}}}{2\omega}a_{11,22} && \text{as} \quad f \to -\infty, \\ \frac{1}{c_{1,2}} &\approx \begin{cases} \dfrac{1}{\chi_{1,2}} + \mathrm{i}\dfrac{E_v^{\frac{1}{2}}}{2\omega}a_{11,22} & \text{(Case I)} \\[2ex] \dfrac{1}{\chi_{2,1}} + \mathrm{i}\dfrac{E_v^{\frac{1}{2}}}{2\omega}a_{22,11} & \text{(Case II)} \end{cases} && \text{as} \quad f \to \infty. \end{aligned}\right\} \tag{4.2}$$

Note that χ_1 and a_{11} in the right-hand side of (4.2) are the variables representing the internal Kelvin wave as $\tilde{f} \to -\infty$, and the continental-shelf wave as $\tilde{f} \to \infty$. Therefore, c_1 in Case I changes properties from the internal Kelvin wave to the continental-shelf wave as $\tilde{f}$ increases. On the other hand, we see that c_1 in Case II does not change properties even if it passes the change-over latitude. This is a striking contrast to the result of the non-frictional model of AR. The frictional eigenvalue c_2 shows the reverse behaviour. Figure 3 shows examples for the two cases. This result has been generalized to a dynamical system of linearly coupled damped oscillators by Grimshaw & Allen (1982).

Since $\lambda_c \approx |E_v^{\frac{1}{2}}/\omega|$ and $\lambda_0 \approx K^{\frac{1}{2}}/\delta_R$, the parameter $\kappa = |E_v^{\frac{1}{2}}\delta_R/K^{\frac{1}{2}}\omega| = 1$ approxi-

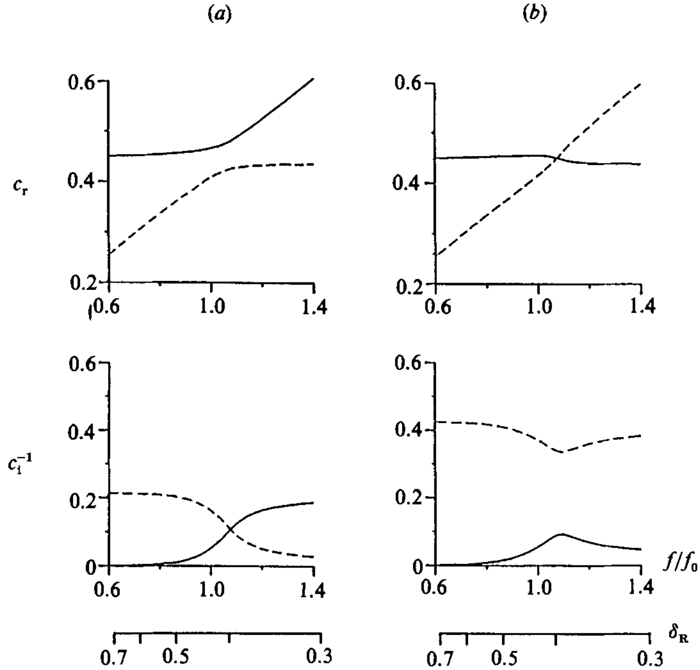

FIGURE 3. Path of the frictional eigenvalues (c_r, the real part; c_i, the imaginary part) with varying $\tilde{f}$. Solid and dashed lines denote the first and second frictional modes, respectively. (*a*) Case I, and (*b*) Case II. Calculations were done using an exponential depth profile $H = H(0)\exp(3.0x)$. Other parameters are $a(0) = 0.1, \delta_R = 0.43$; $|E_v^{\frac{1}{2}}/2\omega| = 0.2$ for Case I, and 0.4 for Case II.

mately distinguishes between Cases I and II. The parameter represents the ratio between the effects of bottom friction and the combined effect of stratification and topography.

4.2. *Frictional eigenfunctions: decoupling of upper- and lower-layer velocities due to bottom friction*

In this subsection, we discuss the effect of bottom friction on the vertical profile of the velocities. For simplicity, we examine the latitude where $a_{11} = a_{22}$.

Frictional eigenfunctions are expressed as

$$\begin{bmatrix} \phi_f^{(1)} & \eta_f^{(1)} \\ \phi_f^{(2)} & \eta_f^{(2)} \end{bmatrix} \propto \begin{bmatrix} R_1^{(1)} & R_2^{(1)} \\ R_1^{(2)} & R_2^{(2)} \end{bmatrix} \begin{bmatrix} \phi^{(1)} & \eta^{(1)} \\ \phi^{(2)} & \eta^{(2)} \end{bmatrix}.$$

where ϕ_f and η_f are the frictional eigenfunctions and the mode is denoted by superscripts (1) and (2) in this subsection.

Here we examine alongshore velocity in the limit of $\kappa \rightarrow \infty$; this requires $K^{\frac{1}{2}} \rightarrow 0$ with δ_R fixed because we have assumed $|E_v^{\frac{1}{2}}/2\omega| < 1$. That is, the coupling between topography and stratification is crucially weak compared with the effects of bottom friction.

At the latitude where $a_{11} = a_{22}$, which is close to the change-over latitude $f = 1$, we can see that

$$R_2^{(1)} = \frac{|a_{12}|}{a_{12}} R_1^{(1)}, \quad R_1^{(2)} = \frac{|a_{12}|}{a_{12}} R_2^{(2)} \quad \text{as} \quad \kappa \rightarrow \infty.$$

Therefore,

$$\begin{bmatrix} v_{f1}^{(1)} & v_{f2}^{(1)} \\ v_{f1}^{(2)} & v_{f2}^{(2)} \end{bmatrix} \propto \begin{bmatrix} 1 & |a_{12}|/a_{12} \\ -|a_{12}|/a_{12} & 1 \end{bmatrix} \begin{bmatrix} v_1^{(1)} & v_2^{(1)} \\ v_1^{(2)} & v_2^{(2)} \end{bmatrix}. \tag{4.3}$$

The subscripts $l = 1, 2$ for v_{fl} and v_l denote the upper- and lower-layer velocities, respectively. As in AR, the amplitude ratio of $\eta^{(j)}$ to $\phi^{(j)}$ is $O(a(0)^{\frac{1}{2}})$ in the neighbourhood of the change-over latitude. For example $\phi^{(j)}$ and $\eta^{(j)}$ for the weak-slope model are

$$\phi^{(j)} = A \sin Mx, \quad M = (b_s/\delta_R)^{\frac{1}{2}},$$

$$\eta^{(j)} = \pm a(0)^{\frac{1}{2}} A \, \mathrm{e}^{-(f/\delta_R)x},$$

where A is the normalization coefficient defined by (2.9). Further, the ratio of the amplitudes can also be estimated from (2.9) since the contribution of the $\eta^{(j)}$ and $\phi^{(j)}$ terms to the integration is of the same order around the latitude. Therefore, we can take $\phi^{(j)}$ and $\eta^{(j)}$ with respect to $a(0)$ as follows:

$$\phi^{(1)} = \phi, \quad \phi^{(2)} = \phi,$$

$$\eta^{(1)} = a(0)^{\frac{1}{2}} \eta, \quad \eta^{(2)} = -a(0)^{\frac{1}{2}} \eta.$$

Now the leading order of the alongshore velocity becomes

$$\left.\begin{aligned} v_1^{(1)} &= a(0)^{\frac{1}{2}} a^{-1} \eta_x, & v_2^{(1)} &= a(0)^{\frac{1}{2}} \phi_x, \\ v_1^{(2)} &= -a(0)^{\frac{1}{2}} a^{-1} \eta_x, & v_2^{(2)} &= a(0)^{\frac{1}{2}} \phi_x. \end{aligned}\right\} \tag{4.4}$$

Further,

$$|a_{12}|/a_{12} = -1 \tag{4.5}$$

because

$$a_{12} = -\int_0^\infty \frac{1}{H^2} (f\phi_x)^2 \, \mathrm{d}x + O(a(0)) < 0.$$

Substituting (4.4) and (4.5) into (4.3), we find

$$\begin{bmatrix} v_{f1}^{(1)} & v_{f2}^{(1)} \\ v_{f1}^{(2)} & v_{f2}^{(2)} \end{bmatrix} \propto \begin{bmatrix} a^{-1} a(0)^{\frac{1}{2}} \eta_x & 0 \\ 0 & a(0)\phi_x \end{bmatrix}. \tag{4.6}$$

Equation (4.6) shows that the upper- and lower-layer motions are decoupled. In conclusion, the lower-layer velocity of the first frictional mode, representing the internal Kelvin wave, is retarded by bottom friction as discussed in Allen (1984). Further, as a new effect because of the slope, the velocity in the continental-shelf wave shows the bottom-intensified structure when $\kappa \to \infty$.

5. Very low-frequency waves

In this section, we consider VLF motions. The discussion here is not restricted to the vicinity of the change-over latitude (as it was in the previous section) since now $|E_V^{\frac{1}{2}}/2\omega| \gg 1$.

Let us consider a sinusoidal wave such that

$$\begin{bmatrix} \psi \\ h \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \eta(x) \end{bmatrix} \mathrm{e}^{\mathrm{i}\omega(c^{-1}y - t)}. \tag{5.1}$$

For the internal Kelvin wave, the variables are expanded in terms of $2\omega/E_V^{\frac{1}{2}}$ as follows:

$$\bar{\phi} = \bar{\phi}_0 + \frac{2\omega}{E_V^{\frac{1}{2}}} \bar{\phi}_1 + \ldots, \quad \bar{\eta} = \bar{\eta}_0 + \frac{2\omega}{E_V^{\frac{1}{2}}} \bar{\eta}_1 + \ldots, \quad \frac{1}{\bar{c}} = \frac{1}{\bar{c}_0} + \frac{2\omega}{E_V^{\frac{1}{2}}} \frac{1}{\bar{c}_1} + \ldots. \tag{5.2}$$

Substituting (5.1) and (5.2) into (2.4) and (2.5), we obtain the lowest-order equations such as

$$\left[\frac{1}{H^2}(f\bar{\phi}_{0x}+\bar{\eta}_{0x})\right]_x = 0, \tag{5.3}$$

$$\bar{\phi}_0 = 0, \quad f\bar{\phi}_{0x}+\bar{\eta}_{0x} = 0 \quad \text{at} \quad x = 0, \tag{5.4a}$$

$$\bar{\phi}_{0x}, \bar{\eta}_{0x} \to 0 \quad \text{as} \quad x \to \infty, \tag{5.4b}$$

where $H = O(1)$ is assumed. From (5.3) and (5.4), it follows that

$$f\bar{\phi}_0+\bar{\eta}_0 = \text{constant} = \bar{\eta}_0(0). \tag{5.5}$$

Therefore the alongshore velocity of lower layer $v_{0,2}$ is retarded by bottom friction, i.e.

$$v_{0,2} = \frac{H_1}{H}(f\bar{\phi}_{0x}+\bar{\eta}_{0x}) = 0. \tag{5.6}$$

The expansion $O(2\omega/E_{\mathrm{V}}^{\frac{1}{2}})$ gives the phase speed and the wave structure of the waves as follows:

$$\bar{\eta}_0 = \eta(0)\,\mathrm{e}^{-fx/\delta_{\mathrm{R}}}, \quad \bar{\phi}_0 = \bar{\eta}_0(0)\,(1-\mathrm{e}^{-fx/\delta_{\mathrm{R}}}), \tag{5.7}$$

$$\bar{c}_0 = -\frac{H_2(0)}{H(0)}\delta_{\mathrm{R}}. \tag{5.8}$$

A detailed derivation is given in the Appendix. As seen in (5.7) and (5.8), this expansion represents the internal Kelvin wave. Since $\bar{c}_0$ is real, the damping rate of the waves is $O(2\omega/E_{\mathrm{V}}^{\frac{1}{2}})$. That is, the typical decay length L_{K} is of the order of

$$L_{\mathrm{K}} = O\left(\frac{1}{k}\frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}\right) \gg \frac{1}{k}, \quad k = \frac{\omega}{\bar{c}_0}.$$

This implies that the VLF baroclinic wave can travel far from the origin because the lower-layer velocity is retarded by bottom friction.

Next, we consider the expansion

$$\hat{\phi} = \hat{\phi}_0+\frac{2\omega}{E_{\mathrm{V}}^{\frac{1}{2}}}\hat{\phi}_1+\ldots, \quad \hat{\eta} = \hat{\eta}_0+\frac{2\omega}{E_{\mathrm{V}}^{\frac{1}{2}}}\hat{\eta}_1+\ldots, \quad \frac{1}{\hat{c}} = \frac{E_{\mathrm{V}}^{\frac{1}{2}}}{2\omega}\frac{1}{\hat{c}_0}+\frac{1}{\hat{c}_1}+\ldots. \tag{5.9}$$

The lowest-order equations then become

$$\left[\frac{1}{H^2}(f\hat{\phi}_{0x}+\hat{\eta}_{0x})\right]_x+\frac{\mathrm{i}}{\hat{c}}\frac{H_x}{H^2}(f\hat{\phi}_0+\hat{\eta}_0) = 0, \tag{5.10}$$

$$\hat{\phi}_0 = 0, \quad \frac{a}{H}(f\phi_{0x}+\hat{\eta}_{0x})+\frac{\mathrm{i}}{\hat{c}_0}\hat{\eta}_0 = 0 \quad \text{at} \quad x = 0, \tag{5.11a}$$

$$\hat{\phi}_{0x}, \hat{\eta}_{0x} \to 0 \quad \text{as} \quad x \to \infty. \tag{5.11b}$$

The equation (5.10) and the boundary conditions (5.11) yield an eigenvalue problem. There is an infinite set of eigenvalues $\hat{c}_{0n}$ and eigenfunctions $(\hat{\phi}_{n0}, \hat{\eta}_{n0})$. The orthogonality condition becomes

$$\int_0^\infty \frac{H_x}{H^2}(f\hat{\phi}_{0n}+\hat{\eta}_{0n})(f\hat{\phi}^*_{0m}+\hat{\eta}^*_{0m})\,\mathrm{d}x+\frac{\hat{\eta}_{0n}(0)\,\hat{\eta}^*_{0m}(0)}{a(0)H(0)} = \delta_{nm}, \tag{5.12}$$

where the asterisk denotes the complex-conjugate variables. Associated with this, we may obtain

$$\left(\frac{1}{\hat{c}_0}+\frac{1}{\hat{c}_0^*}\right) = 0.$$

Therefore, $\hat{c}_0$ is purely imaginary, i.e. it represents the damping rate. When (5.10) and (5.11) are multiplied by $(f\hat{\phi}_{0n}^* + \hat{\eta}_{0n}^*)$, the damping rate becomes

$$\frac{1}{\hat{c}_0} = -\mathrm{i}\int_0^\infty \frac{1}{H^2}|f\hat{\phi}_{0nx}+\hat{\eta}_{0nx}|^2\,\mathrm{d}x. \tag{5.13}$$

Therefore, the typical wavelength L_{S} is

$$L_{\mathrm{S}} = \frac{\hat{c}_0}{E_{\mathrm{v}}^{\frac{1}{2}}} = O(E_{\mathrm{v}}^{-\frac{1}{2}}).$$

This corresponds to the damping length of the continental shelf wave.

The vorticity equation at $O(2\omega/E_{\mathrm{v}}^{\frac{1}{2}})$ gives the solution (the derivation is presented in the Appendix)

$$\hat{c}_{1n} = -\frac{f}{\displaystyle\int_0^\infty \left[\frac{1}{H}\left(|f\hat{\phi}_{0nx}|^2+\frac{|\hat{\eta}_{0nx}|^2}{a}\right)+f^2\frac{|\hat{\eta}_{0n}|^2}{SH_1^2}\right]\mathrm{d}x}. \tag{5.14}$$

We can see that $\hat{c}_{1n}$ is real. Therefore, this represents the phase speed of the continental shelf wave.

Now we have derived the eigenvalues $\bar{c}$ and $\hat{c}$, which correspond to the internal Kelvin and the continental-shelf waves, respectively, and which are strongly modified by bottom friction. Since the ratio of L_{K} to L_{S} is $O(E_{\mathrm{v}}/4\omega^2)$, we cannot find a latitude where $\bar{c} = \hat{c}$ as long as $|2\omega/E_{\mathrm{v}}^{\frac{1}{2}}| \ll 1$ (note that $\hat{c}$ and $\bar{c}$ are complex). That is, we do not need another expansion at the change-over latitude, where the phase speeds (the real part of $\hat{c}$ and $\bar{c}$) coincide with each other; the system shows Case II-type behaviour. In conclusion, the results obtained in §4 are valid for wide parameter range of $|E_{\mathrm{v}}^{\frac{1}{2}}/2\omega|$ even though we assumed $|E_{\mathrm{v}}^{\frac{1}{2}}/2\omega| \ll 1$ in that section.

Further, the analysis of this section is not restricted to the neighbourhood of the change-over latitude. Therefore, we expect that the surface-intensified internal Kelvin waves due to bottom friction are likely to exist in the midlatitudes. In the next section, we suggest some evidence of baroclinic VLF motions in the midlatitudes from observations.

6. Summary and discussion

We have investigated an eigenvalue problem for 'frictional' coastal trapped waves using a two-layer model including shelf and slope topography. We used an f-plane model and have obtained the relationship between the eigenvalues (the phase speed and the damping rate) and the latitude. Further, modifications in the vertical profile of velocities due to bottom friction have been found.

It is one of the important results that the bottom friction considerably modifies the vertical profile of velocities of the VLF coastal trapped waves; the internal Kelvin (continental-shelf) wave attains a surface- (bottom-) intensified structure due to bottom friction. For the internal Kelvin wave, in particular, the surface-intensified structure causes a decrease in damping.

Further, the phase speed and the damping rate around the change-over latitude in the frictional model show different behaviour from those in non-frictional models; for short-period waves, the wave properties change from one wave to another around the change-over latitude (Case I), while for the VLF waves, however, the properties do not change (Case II). This means that the phase speeds intersect at the change-over latitude in Case II.

In the following, we examine κ in a real ocean. We see from (2.10) that

$$K^{\frac{1}{2}} \approx \frac{c_1 - c_2}{\delta_{\mathrm{R}}}\bigg|_{f=1} \approx \frac{c_1^* - c_2^*}{\frac{1}{2}(c_1^* + c_2^*)}\bigg|_{f^* f_c^*},$$

which is $O(1)$, and also $\delta_{\mathrm{R}} = O(1)$ in the neighbourhood of the change-over latitude in actual environments. Therefore, κ is approximated by $|E_{\mathrm{V}}^{\frac{1}{2}}/\omega|$. In §5, further, $|E_{\mathrm{V}}^{\frac{1}{2}}/2\omega|$ was shown to be a key parameter which reflects the effects of bottom friction at any latitude.

Weather-band frequency phenomena have been observed in various coastal regions such as those off Oregon, Peru and Australia (e.g. Kundu & Allen 1976; Romea & Smith 1983; Freeland *et al.* 1986). When we choose the frequency ω as $10^{-5}\ \mathrm{s}^{-1}$, the resistance coefficient r as $10^{-4}\ \mathrm{m\ s}^{-1}$, and the typical depth D as 100 m, we obtain

$$\kappa < 1.$$

This indicates that the weather-band frequency phenomena are characterized by Case I. Therefore, the waves do not travel very far from the origin (O(1000 km)). This result corresponds to observations and models (e.g. Enfield & Allen 1983; Battisti & Hickey 1984).

On the other hand, κ for the VLF waves such as intraseasonal frequency (e.g. Spillane, Enfield & Allen 1987) $[\omega = O(10^{-6}\ \mathrm{s}^{-1})]$ and annual frequency $[\omega = O(10^{-7}\ \mathrm{s}^{-1})]$ fluctuations is

$$\kappa \approx O(1) - O(10).$$

These indicate that the VLF waves are characterized by Case II. Therefore, the baroclinic waves can travel far from their origin. For example, Huyer & Smith (1985) found that the signals caused by the 1982/83 El Niño off the coast of Oregon had arrived by an oceanic path; it led the associated atmospheric disturbances by two or three months. The offshore scale of the phenomena estimated by the ΔD profile was 30 km; this corresponds with the internal Rossby radius. Further, they estimated the phase speed as 140 km/day. These observations can be understood in terms of internal Kelvin waves which propagate long distances from the Equator. Non-frictional models cannot explain such baroclinic behaviour because these models predict that the motions over the shelf in midlatitudes are barotropic (Clarke & Brink 1985). Therefore, the baroclinic structure is likely to be set up by retarding of the lower-layer velocity.

We have analysed a highly idealized, two-layer model. We need further discussions using continuous-stratification models for it to apply to a real ocean. However, the qualitative results we obtain will be realized in the continuous-stratification models because the bottom boundary condition for them can be written as follows (Clarke & Van Gorder 1986):

$$\frac{\phi_z(x,z)}{N^2} + \frac{\phi_x(x,z)}{f^2} + \frac{H_x}{fc}\phi = -\frac{ir}{\omega}\frac{\mathrm{d}}{\mathrm{d}x}[\phi_x(x,z)]_{z=-h},$$

where $(\mathrm{d}/\mathrm{d}x)[\,]_{z=-h}$ is the cross-shelf derivative along the bottom topography. Therefore, if $r/\omega \gg 1$, we obtain

$$-\frac{r}{\omega}\frac{\mathrm{d}}{\mathrm{d}x}[\phi_x]_{z=-h} = 0,$$

for the lowest-order equation. Since $\phi_n \to 0$ as $x \to \infty$, the alongshore velocity $v = \phi_x$ approaches zero over the shelf as $|\omega| \to 0$.

We have used an f-plane model. Therefore, we cannot apply this model directly to the poleward propagation of the coastal trapped waves since changes in the Coriolis parameter should then be considered. The following two effects may be missed by an f-plane model: (i) the non-slowly varying effect for f, which is briefly mentioned in the introduction; (ii) the energy leakage from the coastal region to the ocean interior as Rossby waves.

For a short wave in which f varies slightly over one longshore wavelength, the properties of the propagating waves change along with the eigenvalues which continuously depend on f (Grimshaw 1977; AR). Therefore, the poleward propagation of the weather-band coastal trapped waves is probably characterized by Case I. That is, the internal Kelvin wave originating near the Equator changes its form to a continental-shelf wave in midlatitudes and cannot propagate far from the origin. If the wavelength is longer than the typical scale of changes in f, however, the waves tend to retain their properties as they cross the change-over latitude (e.g. Suginohara 1981). Therefore, both the non-slowly varying effect and the behaviour of the eigenvalues in Case II tend to allow the waves to retain their characteristics as they travel across the change-over latitude. These discussions are presented in a forthcoming paper by the present authors, and Mitsudera (1986).

The f-plane model provides only for trapped waves. If the frequency is very low, however, energy leakage due to Rossby waves becomes important. Grimshaw & Allen (1988) have estimated the critical frequency ω_c between trapped and leaky internal Kelvin waves and have found that the waves of period less than 200 days are coastally trapped in almost all regions along the North American Continent except for the Equatorial region. Therefore, we expect the analysis in §5 to be useful for the waves with $2\pi/\omega_c \leqslant 200$ days in midlatitudes.

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Appendix. Derivation of eigenvalues and eigenfunctions when $|E_V^{\frac{1}{2}}/2\omega| \gg 1$

A.1. *The internal Kelvin wave*

The $O(2\omega/E_V^{\frac{1}{2}})$ equations and boundary conditions become

$$\left[\frac{1}{H^2}(f\bar{\phi}_{1x}+\bar{\eta}_{1x})\right]_x - \mathrm{i}\left[\left(\frac{\bar{\phi}_{0x}}{H}\right)_x + \left(\frac{1}{\bar{c}_0 H}\right)_x \bar{\eta}_0(0)\right] = 0, \qquad (\mathrm{A}\,1a)$$

$$\left[\frac{1}{H^2}(f\bar{\phi}_{1x}+\bar{\eta}_{1x})\right]_x - \mathrm{i}\left[\left(\frac{\bar{\eta}_{0x}}{aH}\right)_x - \frac{f^2\bar{\eta}_0}{SH_1} + \left(\frac{1}{\bar{c}_0 H}\right)_x \bar{\eta}_0(0)\right] = 0, \qquad (\mathrm{A}\,1b)$$

$$\bar{\phi}_1 = 0, \quad \frac{a}{H}f(f\bar{\phi}_{1x}+\bar{\eta}_{1x}) - \mathrm{i}\left(\bar{\eta}_{0x} - \frac{f}{\bar{c}_0}\bar{\eta}_0\right) = 0 \quad \text{at} \quad x = 0, \qquad (\mathrm{A}\,2a)$$

$$\bar{\eta}_{1x}, \bar{\phi}_{1x} \to 0 \quad \text{as} \quad x \to \infty, \qquad (\mathrm{A}\,2b)$$

where (5.5) is used. From (A 1*a*, *b*) and (5.5), it follows that

$$\bar{\eta}_{0xx} - \frac{f^2}{SH_1}\bar{\eta}_0 = 0. \tag{A 3}$$

Therefore, considering the boundary conditions in (5.4), we may obtain

$$\bar{\eta}_0 = \bar{\eta}_0(0)\,\mathrm{e}^{-fx/\delta_\mathrm{R}}, \quad \bar{\phi}_0 = \bar{\eta}_0(0)\,(1-\mathrm{e}^{-fx/\delta_\mathrm{R}}). \tag{A 4}$$

Further, integration of (A 1*a*) gives

$$\mathrm{i}\bar{\phi}_{0x} = \frac{1}{H}(f\bar{\phi}_{1x}+\bar{\eta}_{1x}), \tag{A 5}$$

where the integration constant is derived using (A 2*b*). From (A 2*a*), (A 4) and (A 5), we obtain

$$\bar{c}_0 = -\delta_\mathrm{R}\frac{H_2(0)}{H(0)}. \tag{A 6}$$

A.2. *The continental shelf wave*

A derivation of the $O(1)$ solution is presented in the text. At $O(2\omega/E_\mathrm{V}^{\frac{1}{2}})$, the equations and corresponding boundary conditions become

$$\left[\frac{1}{H^2}(f\hat{\phi}_{1n}+\hat{\eta}_{1n})_x\right]_x + \mathrm{i}\frac{H_x}{\hat{c}_0 H^2}(f\hat{\phi}_{1n}+\hat{\eta}_{1n}) = \mathrm{i}\left[\left(\frac{\hat{\phi}_{0nx}}{H}\right)_x - \frac{H_x}{\hat{c}_1 H^2}(f\hat{\phi}_{0n}+\hat{\eta}_{0n})\right] \tag{A 7a}$$

$$= \mathrm{i}\left[f^{-1}\left(\frac{\hat{\eta}_{0nx}}{aH}\right)_x - \frac{f\hat{\eta}_{0n}}{SH_1^2} - \frac{H_x}{\hat{c}_1 H^2}(f\hat{\phi}_{0n}+\hat{\eta}_{0n})\right], \tag{A 7b}$$

$$\hat{\phi}_{1n} = 0, \quad \mathrm{i}\frac{1}{\hat{c}_{0n}}\hat{\eta}_{1n} + \frac{a}{H}(f\hat{\phi}_{1n}+\hat{\eta}_{1n}) = \mathrm{i}\left(f^{-1}\hat{\eta}_{0nx} - \frac{1}{\hat{c}_0}\hat{\eta}_{0n}\right) \quad \text{at} \quad x = 0, \tag{A 8a}$$

$$\hat{\phi}_{1nx}, \hat{\eta}_{1nx} \to 0 \quad \text{as} \quad x \to \infty. \tag{A 8b}$$

From (A 7*a*, *b*), it follows that

$$\left(\frac{\hat{\phi}_{0nx}}{H}\right)_x = \mathrm{i}\left[f^{-1}\left(\frac{\hat{\eta}_{0nx}}{aH}\right)_x - f\frac{\hat{\eta}_{0n}}{SH_1^2}\right]. \tag{A 9}$$

Therefore, the lowest-order eigenfunction is determined by (5.10) and (A 9). Equation (A 7*a*) is then multiplied by $f\hat{\phi}_{0n}^*$ and (A 7*b*) by $\hat{\eta}_{0n}^*$. These are combined with two additional equations obtained by multiplying the complex conjugate of (5.10) by $f\hat{\phi}_{1n}$ and by $\hat{\eta}_{1n}$. The same procedure is applied to the boundary conditions. Since $\hat{c}_0$ is purely imaginary, we obtain (5.14).

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